MATH3210 - SPRING 2022 - SECTION 001

CHAPTERS 5 AND 6 - REVIEW SHEET

How to use this review sheet

On the exam or problems, you may use any of the definitions and theorems stated on the review sheet, *unless you are explicitly asked to prove a theorem listed here.* Any unnamed theorem you may use without citing. If you use a named theorem, cite that theorem by name when invoking its conclusions. Please don't hesitate to ask questions. I've proofread this, but typos may still lurk!

1. Integration

Definition 1. Let $I = [a, b]$ be a closed interval. A *partition* of I is a finite set $\mathcal{P} = \{a = x_0, x_1, \ldots, x_n = b\}$, which are thought of as "cut points" for breaking up I into smaller subintervals. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions, we say that $\mathcal{P}$ is refined by $\mathcal{Q}$, or that $\mathcal{Q}$ refines $\mathcal{P}$, if $\mathcal{P} \subset \mathcal{Q}$ (ie, $\mathcal{Q}$ cuts up the interval at the points of $\mathcal{P}$, and maybe more).

Definition 2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $\mathcal{P}$ be a partition of $[a, b]$. Then define

$$U(f, \mathcal{P}) = \sum_{k=0}^{n-1} M_i(x_{i+1} - x_i), \quad \text{where } M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$$

$$L(f, \mathcal{P}) = \sum_{k=0}^{n-1} m_i(x_{i+1} - x_i), \quad \text{where } m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

$U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ are the *upper and lower sum approximations* for f. Then define

$$\overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(f, \mathcal{P}) \qquad \underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(f, \mathcal{P})$$

We say that f is *integrable* if $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$

Theorem 1. *Let $I = [a, b]$, $\mathcal{P}$ and $\mathcal{Q}$ be partitions of I such that $\mathcal{Q}$ refines $\mathcal{P}$, and $f : I \to \mathbb{R}$ be a bounded function. Then*

$$L(f, \mathcal{P}) \le L(f, \mathcal{Q}) \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le U(f, \mathcal{Q}) \le U(f, \mathcal{P}).$$

Theorem 2. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. The following are equivalent:*

- *f is integrable*
- *For any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of I such thata $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$*
- *There exists a sequence of partitions $\mathcal{P}_n$ of I such that $U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \to 0$*
- *There exists a real number A and a sequence of partitions $\mathcal{P}_n$ of I such that $U(f, \mathcal{P}_n) \to A$ and $L(f, \mathcal{P}_n) \to A$.*

Theorem 3. *If $f : [a, b] \to \mathbb{R}$ is monotone (either increasing or decreasing), then f is integrable.*

Theorem 4. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable.*

The following theorem is informally stated and has more precise statements in the book. It important to understand when you can use it, but the formal statements are less important:

Theorem 5.

- *Integration is linear (ie, $\int f+g=\int f+\int g$, and $\int cf = c\int f$ whenever f and g are integrable, and $c\in\mathbb{R}$).*
- *Integration is order-preserving (ie, if f and g are integrable and $f(x)\le g(x)$ for all x, then $\int f\le\int g$).*
- *u-substitution and integration by parts are valid integration techniques and can be phrased as formal identities.*
- *If f is integrable on $[a,b]$, f is integrable on any subinterval.*
- *Integrals may be split up along intermediate break points (ie, $\int_a^b f=\int_a^c f+\int_c^a f$).*

Theorem 6 (Fundamental Theorem of Calculus I)**.** *Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that f' exists and is integrable on (a,b). Then*

$$\int_a^b f'(x)\,dx = f(b)-f(a).$$

Theorem 7 (Fundamental Theorem of Calculus II)**.** *Let $f:[a,b]\to\mathbb{R}$ be integrable, and define $F(x)=\int_a^x f(t)\,dt$. Then F is uniformly continuous on $[a,b]$, and differentiable wherever f is continuous. In fact, if f is continuous at $x\in(a,b)$, then $F'(x)=f(x)$.*

2. Infinite Sums and Power Series

Definition 3. A *series* is an expression of the form $\sum_{k=\ell}^{\infty} a_k$, where $\{a_k\}$ a sequence. ℓ is called the *starting index*, and is usually $\ell=0$ or $\ell=1$. A series has an associated sequence of *partial sums* $s_n=\sum_{k=\ell}^{n} a_k$. We say that the series

- *converges* if the sequence of partial sums s_n converges
- *converges absolutely* if the sequence $s_n^+=\sum_{k=\ell}^{n}|a_k|$ converges
- *converges conditionally* if it converges, but does not converge absolutely

Theorem 8 (Term test)**.** *If a series $\sum_{k=\ell}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k=0$.*

Theorem 9 (Comparison test)**.** *Let b_k be a sequence of positive numbers.*

- *If $\sum_{k=\ell}^{\infty} b_k$ converges, $\{a_k\}$ and $K\in\mathbb{N}$ are such that $|a_k|\le b_k$ for every $k\ge K$, then $\sum_{k=\ell}^{\infty} a_k$ converges absolutely.*
- *If $\sum_{k=\ell}^{\infty} b_k$ diverges and a_k is a sequence such that $a_k\ge b_k$ for all k, then $\sum_{k=\ell}^{\infty} a_k$ diverges.*

Theorem 10 (Integral test)**.** *Let $f:[1,\infty)\to\mathbb{R}$ is a decreasing, positive function and $a_n=f(n)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{T\to\infty}\int_1^T f(t)\,dt$ converges.*

Definition 4. If $\{f_k\}$ is a sequence of functions defined on a common interval I, the series associated to f_k is the expression $\sum_{k=1}^{\infty} f_k$. We say that $g_n = \sum_{k=1}^{n} f_k$ is the sequence of partial sums, and that the series

- converges (pointwise) if for every $x \in I$, $g_n(x)$ converges
- converges uniformly if g_n converges uniformly

Each of these two modes has the associated notions of pointwise, absolute convergence (if $g_n^+(x) := \sum_{k=1}^{n} |f_n(x)|$ converges to a function) and uniform, absolute convergence (if g_n^+ converges uniformly).

Theorem 11. *If* $\sum_{k=1}^{\infty} f_k$ *converges uniformly to a function* F*, then* $\int_a^b F(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$

Definition 5. If c_k is a sequence of real numbers, the power series associated to c_k centered at $a \in \mathbb{R}$ is the series $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$. The *radius of convergence* of f is defined to be

$$R = \left(\limsup_{k\to\infty} |c_k|^{1/k} \right)^{-1}.$$

Theorem 12. *A power series converges absolutely to an infinitely differentiable function on the interval* $(a-R, a+R)$*, where* R *is its radius of convergence. Furthermore, the convergence is uniform on any closed interval* $[b,c] \subset (a-R, a+R)$*. We also have that:*

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\infty} k c_k (x-a)^{k-1} \\ \int_a^b f(x)\,dx &= \sum_{k=0}^{\infty} \frac{c_k}{k+1} (b-a)^{k+1} \end{aligned}$$

Finally, $f^{(k)}(x) = k! \cdot c_k$.

Theorem 13. *Let* $f : (a-R, a+R) \to \mathbb{R}$ *be* $(n+1)$*-times differentiable, and*

$$p_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

be the n*th order Taylor approximation. Then for every* $x \in (a-R, a+R)$*, there exists* c *between* a *and* x *such that*

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$